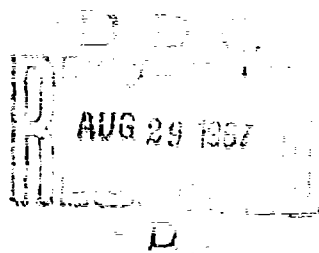RESEARCH PAPER P-338

A HEURISTIC COMPROMISE BETWEEN PROBABILITY
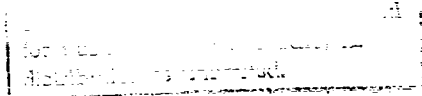
OF SUCCESS AND LIMITED RESOURCES

Murray Turoff

August 1967

AUG 29 1967

RESEARCH PAPER P-338

A HEURISTIC COMPROMISE BETWEEN PROBABILITY

OF SUCCESS AND LIMITED RESOURCES

Murray Turoff

August 1967

ABSTRACT

This paper proposes an analytical technique for modeling by the analyst of a particular decision situation. Evidence supporting the basic proposal, properties of the technique, discussions of its possible applications, and examples are also presented.

The basic problem is a compromise between the probability of success in a single trial versus the total expected number of successes over many trials under the constraint of a limited and uncertain resource. The analogous situation of choosing an operating point for a system which is a compromise between the effectiveness of the system and its efficiency (effectiveness divided by cost) is also discussed.

CONTENTS

*"It takes two of us to discover the truth:
One to utter it and one to understand it."*

- Kahlil Gibran

I. INTRODUCTION

Our goal in this paper is to model a decision point for use by
the analyst, both in the process of analyses and in his communication
with the decision maker. We are in no way attempting to derive an
analytical tool for the making of a decision by the decision maker.
Rather we wish to remove from the analyst the burden of making arbi-
trary decisions in the process of modeling a system. By arbitrary
here we mean a choice that has no intrinsic meaning in the mutual com-
munication between the analyst and the decision maker.

As will be seen, the decision point chosen is an artificial one
in the sense that there is little likelihood that a real-world system
would be operated at the particular decision point. The actual point
is derived as a point of "equality" between two conflicting decision
measures of value. Since a decision maker usually has a preference
(probably indefinable analytically) for one of the particular meas-
ures, this point serves the purpose for the analyst of a boundary on
the region of interest, if he is aware of the decision maker's pref-
erence; or as a reference point about which to examine the behavior
of the system, if he is not aware of the decision maker's preference.
It further serves the purpose of establishing a common point of ref-
erence for both the decision maker and the analyst.

Since the decision process is, by definition, one in which an
optimum choice does not exist in the classical analytical sense, the
technique proposed in this paper cannot be supported in a rigorous
manner. Therefore, the methodology of this paper is to support the
choice of a particular analytical definition based upon a number of

1

interesting properties resulting as a consequence of the choice. In
other words, to present circumstantial evidence, but enough of it to
cast "intuitive" credibility upon the basic premise.

II. THE PROBLEM

We assume the following situation:

(1) For a single occurrence of an event, we are given a proba-
     bility of success (P) which is a _known_ function of some
     amount of resource (n) we choose to invest in the single
     trial. Examples of this are the probability of killing a
     target when we salvo n shots at it or the probability of
     not losing our money in loaning n dollars to an insecure
     business.

(2) P(n) is a monotonic increasing function to some finite
     limit (1 or less) as n goes to infinity; or, P(n) increases
     with n to some maximum value before it falls off.

(3) The total supply of our resource (N) is finite. Therefore,
     the total number of separate independent trials of the
     event that can take place is N/n. Since this is a binomial
     process the total expected number of successes (E) out of
     the N/n trials will be

$$E = \frac{N}{n} P(n) \tag{1}$$

(4) The exact value of our total resources (N) is unknown.
     Through uncertainties in our budget, the ultimate cost of
     the resource or the number of trials we will require, N has
     a significant uncertainty.

(5) Regardless of the uncertainty of N, it is a limited resource
     in the following qualitative sense: If we demand a high
     probability of success in each trial (P(n) $\approx$ 1, n large) we
     expect N/n to be small resulting in a small return in terms

5

of the expected number of successes as in Eq. 1. If, however, we attempt to make n small in order to obtain a maximum expected return, we are presented with a low or unacceptable probability of success in a single trial (P(n) small compared to 1).

The situation as described is depicted in Fig. 1. If we wish a high probability of success in a single trial we are forced to large n. If we want a high number of expected successes we are forced to low values of n. It is these two opposing measures with which we are now faced--opposing in the sense that a preference for one tends to dictate an opposite choice of n. The crucial and perhaps non-real-world assumption we now present is:

(6) Our mythical decision maker wishes to choose a value of n which represents a point of equality between these opposing measures of value. To him there is an equality of importance or value between the individual (single trial) probability of success and the total expected number of successes.
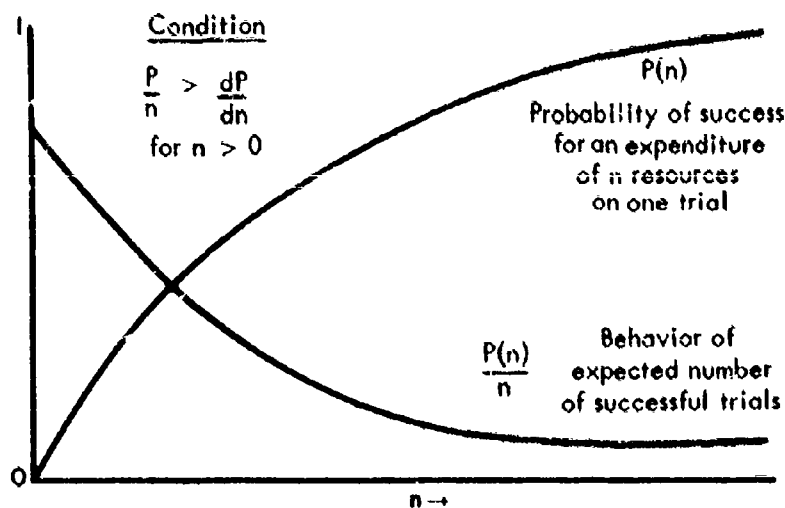

Condition

$\frac{P}{n} > \frac{dP}{dn}$

for n > 0

P(n)
Probability of success
for an expenditure
of n resources
on one trial

$\frac{P(n)}{n}$ Behavior of
expected number
of successful trials

n →

FIGURE 1   Basic Problem

4

The problem as now stated is to attempt to define an analytical criterion for the concept of "equality" in this situation.

The type of probability distribution illustrated in Fig. 1 has no point of maximum expected number of success except at the lowest allowable value of n. Mathematically this condition is characterized by $P/n > dP/dn$. This type of function is our primary concern; however, we will consider distributions exhibiting an internal point of maximum expected successes in a later section of this paper. If, in this situation, the analyst makes an arbitrary choice of a confidence limit to pick a point n, he has produced a point which has no intrinsic meaning communicable between him and the decision maker. Our goal, therefore, is to define a point which has a meaning in terms that are understood by both parties.

III. USEFULNESS

Before proceeding to the actual treatment of the problem, some clarification of the uses for such a technique may provide a proper perspective for the reader.

A point of "equality," "compromise," or "indifference" between these two opposing measures of value would have unique value as a boundary and/or reference point in the decision process. If n_0 is the actual point of equality, we have the following results.

Total winnings more important	$\Rightarrow$	$n \leq n_0$	upper bound on n
Probability of success more important	$\Rightarrow$	$n \geq n_0$	lower bound on n
Unknown relative importance	$\Rightarrow$	$n \approx n_0$	reference point

Therefore, with binary value judgment (which measure has greater importance) the analyst can considerably reduce the range of consideration for the variables in the problem. While the analyst usually has no exact knowledge of the relative weights a decision maker implies to these two properties, he can usually gather from the situation which measure is of greater importance. In other words, it is always easier to make an accurate binary decision than a continuous one. This in no way, however, removes the responsibility of the analyst to examine sensitivity over the range of n once it is established by the point of equality. As an example, in some military situations the individual probability of success is more important while, unfortunately, in some commercial situations the total return is more important than the probability of successfully satisfying each customer.

Frequently, in military or commercial situations a requirement to model a competitor's system is put upon the analyst and not the decision maker. The purpose, of course, is to test the effect of possible alternatives our decision maker may have upon the competitor's actions. In this light, let us imagine the following scenario:

(1) We wish to deter a competitor from making a decision to begin a certain action involving many trials of a certain type.

(2) His probability of success in any trial is given by a function $P'n, x_i)$ where $\{x_i\}$ is a set of variables under our control.

(3) We have the knowledge that his total supply of the resource N is limited or expensive.

(4) We also know that he values individual probability of success somewhat _more_ than total expected number of successes.

Therefore, if we know the point of equality of importance between these criteria (say n_0), our goal is to set the values of x_i such that $P(n_0, x_i)$ will be costly for him to obtain. In other words, we wish to minimize the possibility of his being able to afford the point where he would consider the decision to undertake the action.

We might contrast this approach with the arbitrary choice of a defense conservative limit: a choice of n such that P(n) is so low that everyone has agreed that only an insane person would exercise action at this point. The usual problem with this limit is that practically any choice of x_i by our decision maker shows up as a very expensive proposition in order to negate the taking of the action. The opposing limit, offense conservative, also imposes certain difficulties for the decision maker. By establishing a very high value for P(n), the total resource (N) ultimately required for exercising the action becomes so costly that there is a large degree of insensitivity in the various choices of x_i that might be exercised by our own decision maker. In other words, practically any inexpensive choice will

negate the taking of the action. One way of thinking of this point of equality is as a compromise between the offense-defense conservative boundaries.

Very often the analyst must set up a base design which is used as a reference point. It would probably be helpful to the decision maker if such a point were determined upon an analytical model rather than some arbitrary choice of the analyst. Reproducibility of a reference point among different analysts and decision makers should be advantageous to both.

In some instances the analyst is faced with the question of having to inform the decision maker quickly as to whether or not there is any possibility that a particular system is within the realm of reasonable cost. Under the constraint that the individual probability of success is more important, the costing of the system at n_0 usually provides the lower bound of the cost. Therefore, this approach provides a quick, back-of-envelope-type indication for deciding whether further study is warranted.

In the computer simulation of very complex systems, more often than not the analyst must provide a multitude of probability factors to simulate hardware that is undeveloped. In many instances it is impossible from a time-and-effort standpoint to undertake a complete sensitivity analysis of all these parameters; nor does the decision maker always have the time to dig into the model to sufficient depth to determine if the arbitrary choices by the analyst conformed to his own intuition. While this is unfortunate from an academic viewpoint, it appears to be the real-world situation. This approach provides a common understandable ground rule for the analyst and the decision maker to establish a base or reference case from which the excursions can be conducted. Furthermore, applying the ground rule to two separate complex systems which are candidates for the same job provides a common point of comparison.

IV. TREATMENT

What follows is a set of arguments and analogies of a rather diverse nature, all of which lead to a unique analytical definition for establishing a specific point n_0 in the tradeoff between number of successes ($NP(n)/n$) and the individual probability of success ($P(n)$). Some of these arguments may be more enlightening or satisfying to the reader than others. In addition, some of the properties exhibited could have been taken as assumptions to define n_0. We have not chosen to do so in order to attempt to establish a more general meaning for the results. It is the sum total of all the arguments which the author would ask the reader to consider in his evaluation of the premise.

A. CHOICE OF A VALUE OR UTILITY FUNCTION

Let us hypothesize that there exists a function $H(f(x), g(x))$ which takes on a maximum value at a point x_0 and expresses "equality" or "indifference" between the two measures of value represented by the functions $f(x)$ and $g(x)$ at that point. In the neighborhood of x_0 the functions f and g are considered to be opposing functions of x: one increases while the other decreases for any small change in x about x_0. In our particular problem f and g represent the individual probability of success and the total number of expected success. The concept of the existence of a value or utility function is not new to either the fields of psychology[1,2] or economics.[3]

    (a) This premise of the existence of such a value function is assumed here to mean it should be analytic in f and g:

9

$$H\left(f(x),\ g(x)\right) = \sum_{i,j} a_{ij} f^i g^j \qquad (2)$$

This ensures we are considering differentiable functions with a unique measure for the value function at each point x.

(b) We further hypothesize that impartiality or indifference leads us to the following assumption. If

$$\frac{dH}{dx}\left(f(x),\ g(x)\right) = 0 \qquad (3)$$

defines a point x_0, then

$$\frac{dH}{dx}\left(\alpha f(x),\ \beta g(x)\right) = 0$$

defines the same point x_0 for any arbitrary constants (non-zero) α and β. This condition expresses the requirement that our results should be insensitive to any arbitrary change of scale or dimension in the functions f and g. In our particular case it implies that we seek a result insensitive to the value of N and its inherent uncertainties.

(c) Since the value function must exhibit impartiality, indifference, or equality between f and g, in addition we have the assumption that

$$H(f,\ g) = H(g,\ f) \qquad (5)$$

Equations 3 and 4 applied to Eq. 2 result, respectively, in:

$$\sum a_{ij}\ \frac{d}{dx}\left(f^i g^j\right) = 0 \qquad (6)$$

$$\sum \alpha^i \beta^j a_{ij}\ \frac{d}{dx}\left(f^i g^j\right) = 0 \qquad (7)$$

Subtracting the two equations we see that the only way they can both be satisfied at an arbitrary point x_0 for any arbitrary values of α and β is if

10

$$a_{ij} = 0$$

or (8)

$$\frac{d}{dx}(f^i g^j)\Big|_{x_0} = 0$$

Therefore, our value function must be of the form

$$H(f, g) = C_0 f^{\gamma_1} g^{\gamma_2} \tag{9}$$

Our assumption of indifference (Eq. 5) now implies

$$\gamma_1 = \gamma_2 = \gamma \tag{10}$$

or that

$$H(f, g) = H(fg) = C_0(fg)^\gamma \tag{11}$$

or

$$\frac{d}{dx}(fg)\Big|_{x_0} = 0 \tag{12}$$

as a further observation, the results will be insensitive to any functional change of x to another variable c

$$x = x(c) \tag{13}$$

since solving for the tradeoff value of c will provide the same value of x_0 through the use of Eq. 13. Therefore, the same results are obtained whether we are talking about the number of shots or the cost of each shot, or whether we are talking about man-hours or the cost of a man-hour.

(d) Finally, if we assume that under the condition of limited resources we may consider applying two independent systems to the same job, and because of the overall limit on resources they do not overlap in effectiveness, we should then recognize that the value function is additive

11

$$H(f_1 g_2) + H(f_2 g_2) = H(f_1 g_1 + f_2 g_2) \tag{14}$$

which implies $\gamma = 1$.

The result, in terms of our original problem, is that

$$\boxed{\text{Value Function} = \frac{NP^2(n)}{n}} \tag{15}$$

or that n_0 is defined by the equation

$$\boxed{\frac{dP}{dn} = \frac{P}{2n}} \tag{16}$$

Occasionally, in trading off two opposing functions, a mean of some sort (average, geometric, harmonic) is taken as a value function to maximize. This result, as expressed by Eq. 15, is equivalent to using the geometric mean as the value function. In general, the geometric mean occupies a middle-of-the-road position between the arithmetic and harmonic means.

Arithmetic Mean $\geq$ Geometric Mean $\geq$ Harmonic Mean

The geometric mean is not as sensitive as the arithmetic mean to a high value or as sensitive as the harmonic mean to a low value.

The conclusion reached here is that if a compromise point exists between two functions which is insensitive to arbitrary or uncertain weights applied to the functions then it can be found by maximizing the product of the two functions.

It is of interest to note that assumptions of analyticity or regularity usually applied in constructing value functions for probalistic situations (see for example: Human Judgments and Optimality, edited by M. Shelly and G. Bryan, pp. 189-191) disallow use of the standard deviation as an explicit term in the value function except perhaps as an approximation to a general but unknown value function in the neighborhood of the solution.

12

B. COMPROMISE BETWEEN EXPECTED VALUE AND VARIANCE

While the expected value in the binomial distribution is

$$E = \frac{N}{n} P(n) \tag{17}$$

the variance is expressed by

$$V = \frac{N}{n} P(n) \left(1 - P(n)\right) \tag{18}$$

In the type distribution diagramed in Fig. 1, the variance is also a decreasing function of n for a constrained resource N; therefore, high expected value also means high variance or low confidence in obtaining the expected value.

One way of handling this is to use standard deviation $\sqrt{V}$ and to set arbitrary confidence limits and maximize the probability within these limits. Quoting from A. C. Aitken in his book <u>Statistical Mathematics</u>: "Modern usage is tending more and more to treat variance itself, rather than standard deviation, as a suitable measure of dispersion." Taking this statement as gospel and utilizing the concept of marginal return, let us assume that a compromise between expected value and the confidence in obtaining the expected value is that point where the marginal return in expected value is equal to the marginal return in variance:

$$\boxed{\frac{dE}{dn} = \frac{dV}{dn}} \tag{19}$$

Therefore, at the point n_0 defined by Eq. 19, we are receiving an equal rate of return in both variance and expected value per unit expenditure of the resource n. Since

$$E - V = \frac{N P^2}{n} \tag{20}$$

13

maximizing Eq. 15 is equivalent to Eq. 19 and both result in the condition

$$\frac{dP}{dn} = \frac{P}{2n} \tag{21}$$

The real effect of using Eq. 19 to determine an operating point will probably be clearer to the reader when he examines its effect on some of the probability functions illustrated in the example section of this paper, Sec. VI. As seen there, for n, less than the point defined by Eq. 19 the variance is decreasing at a faster rate than the total expected number of successes. After this point the reverse is true.

A rather pleasant by-product of trading off expected value directly with variance is the additive property of both.

Let us suppose we have two (applies to any number) systems (1 and 2) we wish to apply to the same task. Each utilizes a different resource (x and y) and we have already picked values for x and y based upon our criteria

$$\frac{dE_1}{dx} = \frac{dV_1}{dx} \qquad \frac{dE_2}{dy} = \frac{dV_2}{dy} \tag{22}$$

We may express the overall expected value and variance as

$$E_T = E_1 + E_2$$
$$V_T = V_1 + V_2 \tag{23}$$

Assuming there exists a resource common to both systems (cost) such that we may write

$$x = x(c) \qquad y = y(c) \tag{24}$$

then our new system satisfies

$$\frac{dE_T}{dc} = \frac{dV_T}{dc} \tag{25}$$

14

This may also be interpreted as an additive property for the value function $E - V$. By adding systems we add value, and our criteria of value should apply equally to a single or composite system. This is correct only as long as we are in the limited resource situation and the systems are not overlapping in their effectiveness.

C. EQUALITY OF CONFIDENCE IN THE MACRO AND MICRO VIEWS

Contrary to the usual intuition, if one wishes to ascribe a dimensionality to N/n, then E and V (not the standard deviation $\sqrt{V}$) are of the same units. $P(n)$ by definition is a ratio of two quantities of the same units and therefore is dimensionless. There is no contradiction in using $E - V$ as a value function in opposition to the usual use of $E - \lambda\sqrt{V}$ (λ any constant). It is certainly a no less arbitrary technique.

From the standpoint of units of expectation, however, there is a clear meaning for the value function. The quantity*

$$s = \frac{1}{P(n)} \qquad (26)$$

is the expected number of trials needed to obtain a single success when we expend an amount of resource equal to n on each of the s trials. If we divide s into the expected number of successes (N/n) $P(n)$, we obtain a quantity

$$\frac{\text{expected number of successes}}{\text{expected number of trials for one success}} = \frac{N}{n} P^2 \qquad (27)$$

---

*For a finite number of trials $\left(\frac{N}{n}\right)$, s is actually

$$s = \frac{1 - \left(1 - P(n)\right)^{\frac{N}{n}}}{P(n)}$$

however, in most instances $\left(1 - P(n)\right)^{\frac{N}{n}}$ is essentially zero as compared to one unless the total resource (N) is so limited that expending all of this resource N on only one trial produces a small probability of success. In any case, $1/P(n)$ is always the upper bound on the expected number of trials required for a single success.

15

Therefore, maximizing the value function is, in reality, a simultaneous attempt to maximize the total number of successes while minimizing the number of trials to obtain a success. One may picture this value function as providing an inverse equalization between successes (successful trials) and failures (non-successful trials).

Furthermore, the quantity

$$C = sn \tag{28}$$

is now the expected cost in resource for one successful trial, while the variance in obtaining successes at a cost C is (geometric distribution)

$$V_0 = n(s^2 - s) \tag{29}$$

The maximization of our value function $E - V$ is equivalent to minimizing $C + V_0$

$$n \text{ of max } (E - V) = n \text{ of min } (C + V_0) \tag{30}$$

One may generally show that (1) if it is required that a maximum of some lower confidence limit on the number of successes is required,

$$\text{maximize: } E - a_1 V^{a_2} \tag{31}$$

where a_1 and a_2 are arbitrary constants, and (2) it is required that some minimization on an upper confidence bound on the cost of a success is required,

$$\text{minimize: } C + b_1 V_0^{b_2} \tag{32}$$

where b_1 and b_2 are arbitrary constants. Then the only choice of the constants a_1, a_2, b_1, b_2 which provide the same value of n as solutions to both Eqs. 31 and 32 is

$$a_1 = a_2 = b_1 = b_2 = 1 \tag{33}$$

16

provided a_1 and b_1 are non-zero. Therefore an added interpretation is a compromise between the macro (total number of successes) picture and the micro one (single success) in trying to arrive at an equal partition of confidence in the two views.

To adopt any other form for a confidence limit means that optimizing either one of the two criteria, total successes or cost of a single success, will result in the other being off optimum. While it is certainly true that one cannot simultaneously maximize effectiveness and minimize cost for a given system, we have shown that a particular confidence limit exists which allows maximization of effectiveness at that limit with minimization of cost at the analogous confidence limit along with the result that the solution is insensitive to the total resource.

As a further note, when N/n becomes large, the binomial distribution may be approximated by either the geometric or Poisson distributions depending upon whether P(n) remains finite or goes to zero respectively. Equivalently it may be stated that as E approaches V in value ($E \approx V$) the Poisson distribution is a good approximation (being exact for $E = V$). By maximizing $E - V$ we are attempting to stay away from the Poisson distribution and establish an "equality" of confidence between the macro view (binomial) and the micro view (geometric) by maintaining a reasonably high P(n). It might therefore be concluded that we are attempting to arrive at a point where there is indifference to viewing the process at a macro or micro level. In particular, under this circumstance, if we are able to evaluate the success or failure of each independent trial in turn, it makes no difference in our total return whether we choose a new event (new target or investment) after a failure or continue with a new trial on the same event until a success is obtained. In certain situations this would tend to ensure a certain degree of flexibility in the actual operation of the system.

17

D. A RESPONSE TO UNCERTAINTY

We may rewrite n as

$$n = \frac{N}{T} \tag{34}$$

where N is a measure of our total resource and T is the total number of trials we would like to make with the resource by distributing n on each trial. The expected number of successes is now

$$E = TP\left(\frac{N}{T}\right) \tag{35}$$

Whenever N is an expensive resource a common policy is to try to maintain a constant value of n. It may not be the best policy in all situations, but it is nonetheless frequently employed in many situations. One difficulty in this policy is the inherent uncertainty in N and T, which we will indicate by $|\delta N|$ and $|\delta T|$ where N and T may lie in the bounds

$$N - |\delta N| \leq N \text{ actual} \leq N + |\delta N| \tag{36}$$

$$T - |\delta T| \leq T \text{ actual} \leq T + |\delta T| \tag{37}$$

From elementary error theory the expected uncertainty in n may be written as

$$\frac{\delta n}{n} = \pm \sqrt{\left(\frac{\delta T}{T}\right)^2 + \left(\frac{\delta N}{N}\right)^2} \; {}^{*} \tag{38}$$

Since N is really our resource as compared with T which is probably an opponent's resource, or at least not as well known, we may assume that

$$\frac{|\delta N|}{N} \leq \frac{|\delta T|}{T} \tag{39}$$

---

*This is true for uniform- and Gaussian-type error distributions. See *Introduction to the Theory of Error*, Yardley Beers.

The percent error in estimating T is greater or equal to our relative error in N.

Now, to be conservative we ask what is the worst possible situation. This occurs when we have overestimated our resources by an amount $|\delta N|$ and underestimated the number of trials required by an amount $|\delta T|$ and when equality exists in Eq. 39. In other words, our errors in our resource are just as great as those of our opponent and when we decide to take the action (perform the trials), we find we have less resource and more trials to perform than we expected. We may write this condition as

$$\frac{\delta N}{N} = - \frac{\delta T}{T} \tag{40}$$

To see what happens to our expected return we take the variation of E from Eq. 35.

$$\delta E = \delta T \left[P - n \frac{dP}{dn} \right] + \delta N \frac{dP}{dn} \tag{41}$$

substituting for δN from Eq. 40 we have

$$\delta E = \delta T \left[P - 2n \frac{dP}{dn} \right] \tag{42}$$

If we then require that under a finite uncertainty δT we wish $\delta E = 0$ or that our return or winnings stay constant under the worst uncertainty situation, we have

$$\frac{dP}{dn} = \frac{P}{2n} \tag{43}$$

which is, hopefully, familiar by this time. Therefore we have shown that for first order uncertainties in the system ($\delta T/T \ll 1$) our return E stays constant ($\delta E = 0$) under the worst possible situation ($\delta N/N = - \delta T/T$), with the caveat that our uncertainty in our own resource is no worse than our estimate of our opponent's resource.

19

One may then interpret the planning point n_0 defined by Eq. 43 as a stability point for the expected winnings. As a result of this the minimum total buy of resource one would like to have in this deterrent-type situation is

$$N = n_0 T \qquad (44)$$

with, perhaps, the hope that maybe one can ultimately afford

$$n = |\delta N| + n_0 \left(T + |\delta T| \right) \qquad (45)$$

E. AN ANALOGY WITH PSYCHOLOGICAL THEORY

Quite a bit of experimentation has been carried out in the area of "subjective probabilities." While this subject is not directly analogous to our situation, there are some similarities. One of the current theories in this area is perhaps best summarized in the original statement of the theory.[4]

> The strength of motivation to perform some act
> is assumed to be a multiplicative function of the
> strength of the motive, the expectancy (subjective
> probability) that the act will have as a conse-
> quence the attainment of an incentive, and the
> value of the incentive (winnings):
>
> Motivation = f(Motive x Expectancy x Incentive)

In addition to this description of a value function for deciding to perform an act, it is hypothesized that a similar term describes the motivation for not taking the action which involves the expectancy of failure and the loss associated with this.

In relation to our own problem, the one the psychologists have studied involves an intuitive estimate of the expectancy on the part of the subjects, in opposition to our situation where expectancy is known exactly. Whereas the subjects knew the incentive exactly, our winnings are uncertain due to uncertainty in total resource or number of trials to be made. Even with these differences it would appear analogous to draw the correspondences:

probability of success $\left(P(n)\right)$ corresponds to expectancy

expected success $\left(\frac{N}{n} P(n)\right)$ corresponds to incentive

Since we are principally concerned with questions of designing or evaluating the performance of a physical system as compared to alternate systems, motive in our case is either zero or one. Furthermore, the question of motivation against taking the action is really associated with the actual employment of the system. In commercial situations under limited resources one often cannot do anything about the negative motivation, since if it is present it is probably insensitive to any variables under control. In the military situation there is often no alternative to building a system for a specific job; the only choice is whether to employ it, which often is in the hands of the opponent.

With the realization that our approach concerns the evaluation of possible alternative systems rather than questions of whether or not one should have any system at all (which is really the direct province of the decision maker rather than the analyst), it would appear the value function described in this paper provides a measure of a system in terms of what some psychologists believe is a measure of intuitive value or usefulness by providing a theoretical operating point which maximizes motivation when the system must be employed in an uncertain environment--unknown actual resources or uncertain required use of the system.

Ultimately, as is the case in the physical sciences, the final verification of a theory should rest upon experimental evidence. While the designing of scenarios to test the hypothesis proposed here should not be overly difficult, the problem of obtaining a sufficient set of subjects at the analyst and decision maker levels may be a prohibitive problem for any psychologist interested in examining this question.

21

As a reflection upon the examples provided in Sec. VI, one additional result from this field of psychology is a rough correspondence between $P(n)$ and intuition (for an average subject):[5]

P		intuition
0.7	$\approx$	is certain to
0.6	$\approx$	is likely to
0.55	$\approx$	probably will

F. AN ANALOGY WITH ECONOMIC THEORY

In the theory of demand a measure often employed is the elasticity between price (p) and quantity (q).

$$\text{elasticity} = \alpha = - \frac{p}{q} \frac{dq}{dp} \qquad (46)$$

The use of elasticity as a measure is chiefly justified in that it provides a percentage change rather than absolute change figures. It literally allows the economist to make direct comparisons between apples and oranges.

In this situation pq is a utility function to maximize if the manufacturer has a monopoly and either there is no cost or price is actually profit (relative price) and quantity is a function of this profit or relative price.

$$\text{Utility Function} = pq \qquad (47)$$

The situation in which the elasticity equals one is the point of least sensitivity for this utility function

$$pq \approx \text{constant} \qquad (48)$$

If p and q are some function of a variable x such that a change of x decreases one and increases the other, then an elasticity of one is the point where there will be a minimal change in the product.

22

In our situation we can make a direct comparison in the following manner:

(a) Relative price or profit in our system is equivalent to probability of success*

$$p = P(n) \qquad (49)$$

(b) Quantity is just the expected number of successes and is a function of the profit or relative price

$$q = \frac{N}{n} P(n) \qquad (50)$$

Furthermore, since we are concerned with the evaluation of a specific system for a specific task we are in the analogous situation of having a monopoly--essentially control of setting the operating point of the system.

In Fig. 2 we plot some "demand" curves (P(n) versus P(n)/n) for a particular probability distribution. In opposition to the usual demand curve we have a reversal of curvature with respect to the origin. From the plots it is obvious the point arrived at is essentially a tradeoff between P(n) and P(n)/n in terms of sensitivity of change with respect to each other. This particular type plot applied to a probability situation is quite informative for any probability distribution.

In the situation described here we have one advantage over the economist in that we can often describe the functional dependence of P on n analytically.

---

*In the classical definition of probability as the limit ratio of occurrences to total trials, it would seem to be a relative price rather than an absolute one.
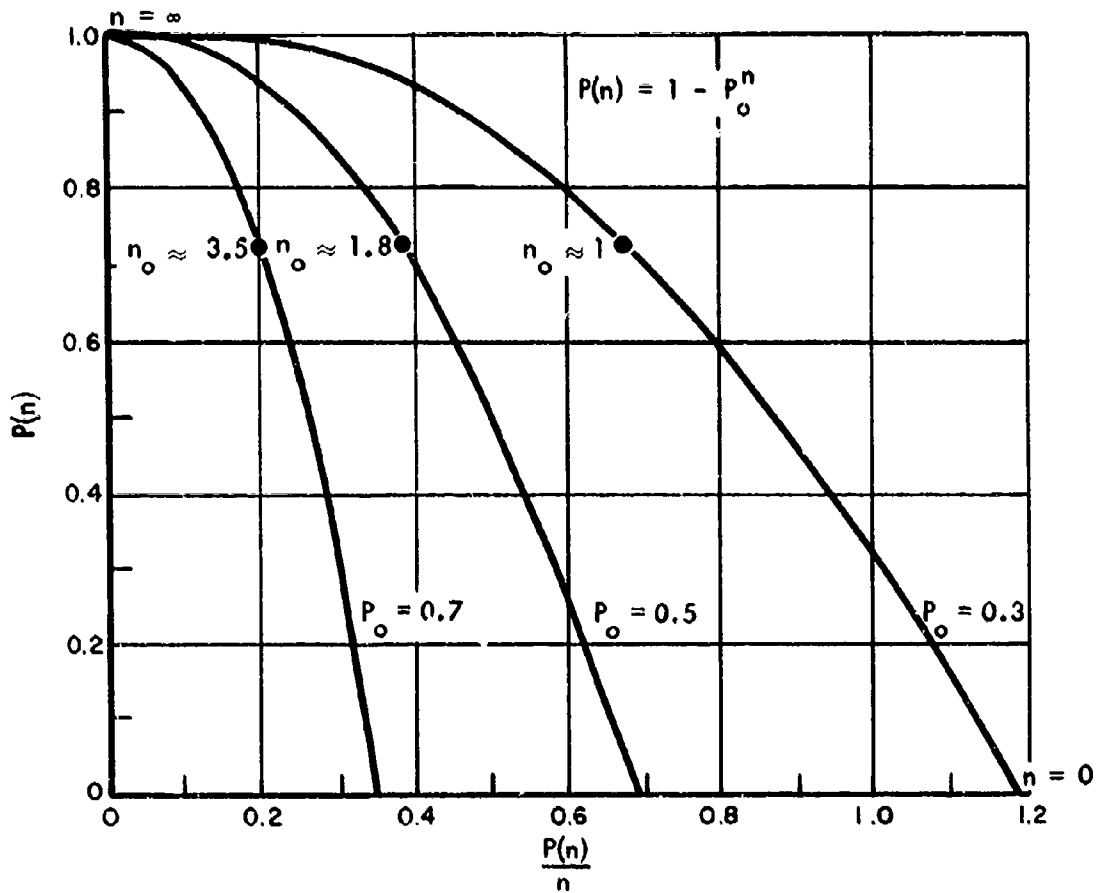
FIGURE 2   Demand Curve Version

24

V. GENERAL OBSERVATIONS

Integrating Eq. 16 we may write down some relations for quickly calculating the behavior about the point n_0 as defined by our criteria:

Macro View	Micro View	
$\dfrac{P(n)}{P(n_0)} = \sqrt{\dfrac{n}{n_0}}$	$\dfrac{s(n)}{s(n_0)} = \sqrt{\dfrac{n_0}{n}}$	(51)
$\dfrac{E(n)}{E(n_0)} = \sqrt{\dfrac{n_0}{n}}$	$\dfrac{C(n)}{C(n_0)} = \sqrt{\dfrac{n}{n_0}}$	(52)
$\dfrac{V(n)}{V(n_0)} = \dfrac{\sqrt{\dfrac{n_0}{n}} - P(n_0)}{1 - P(n_0)}$	$\dfrac{V_c(n)}{V_c(n_0)} = \dfrac{s_0 - \sqrt{\dfrac{n}{n_0}}}{s_0 - 1}$	(53)

It is of interest to note that the probability of success and the cost of obtaining a success behave in the same manner about the point n_0

$$\frac{P(n)}{P(n_0)} \approx \frac{C(n)}{C(n_0)} \tag{54}$$

or equivalently

$$\frac{E(n)}{E(n_0)} \approx \frac{s(n)}{s(n_0)} \tag{55}$$

While we have talked largely of probability of success, we may generalize the result to any measure of effectiveness in the employment of a system. Assuming our effectiveness is a function of cost, we define

25

$$\text{Effectiveness} = \epsilon(c) \left(\lim_{c \to \infty} \epsilon(c) \text{ is finite} \right) \tag{56}$$

$$\text{Efficiency or average return} = \frac{\epsilon(c)}{c}. \tag{57}$$

$$\text{Elasticity} = \frac{\epsilon}{c \frac{d\epsilon}{dc}} - 1 \tag{58}$$

The point where the elasticity between what we choose to call effectiveness and efficiency equals one may be rewritten in the following interesting manner:

$$\frac{1}{c} \frac{d\epsilon}{dc} + \frac{d}{dc} \left(\frac{\epsilon}{c} \right) = 0 \tag{59}$$

With respect to effectiveness the first term is the average return of the marginal return while the second is the marginal return of the average return.

In many systems the most efficient operating point is the one which prescribes a very inexpensive design point (maximum possible ϵ/c) while using overall effectiveness (ϵ) as the criteria leads to a very expensive system. The technique we have discussed would seem to provide a meaningful compromise between overall effectiveness and the efficiency of the system. For any system in which one is more interested in the overall effectiveness, this technique would indicate the minimum investment that should be made if the system is to be built or employed.

We have, of course, now made an intuitive jump from probability of success ($P(n)$) to generalized effectiveness ($\epsilon(c)$). When one considers the inherent uncertainties in cost and its functional relationship to both effectiveness and parameters outside the control of both the decision maker and the analyst, it is usually safer to treat an effectiveness measure with the same consideration one would give to a probability.

One must usually be careful in treating an effectiveness curve with this criterion to ensure that each point on the curve has been optimized for the particular investment represented by the value of c. In some cases this may imply that different regions of c represent completely different systems. However, it will usually be the case that when applying the technique to different effectiveness curves, representing different systems for the same job, the resulting point for any system (c_0 and $\epsilon(c_0)$ for each system) will fall in the region representing the investment level for which that system is best suited, provided that region exists at all for the given system as compared to the others. The exceptions to this usually occur only when the region (investment range) in which a given system is more effective is quite small in comparison to that of the other systems.

In Fig. 3, a visual comparison is made between the proposed techniques (a compromise between ϵ/c and ϵ as opposed to the criterion of maximum expected value or efficiency (ϵ/c)). For those distributions
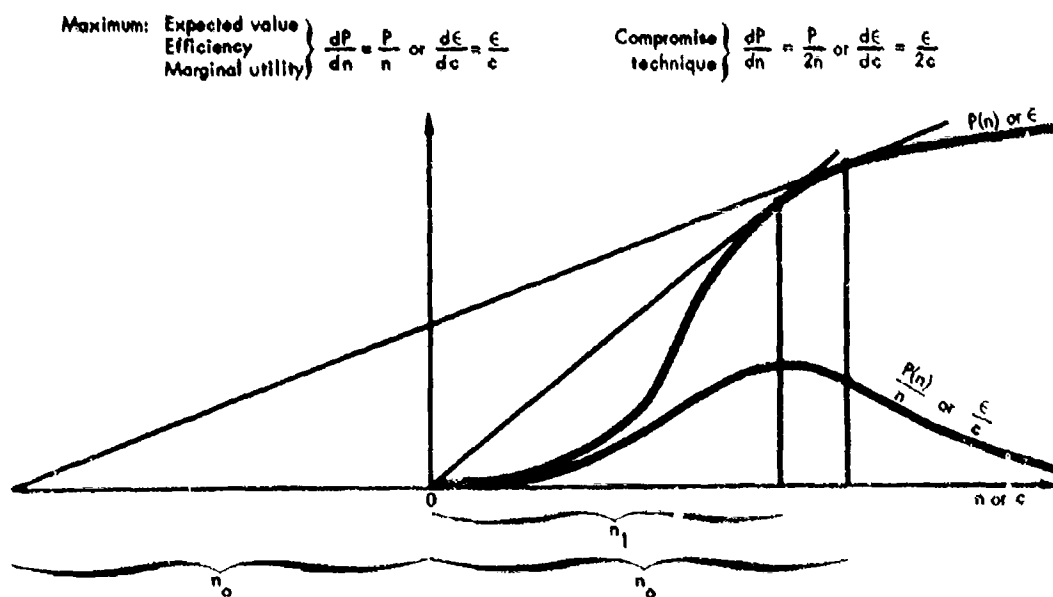

FIGURE 3 Comparison with Marginal Utility or Maximum Expected Value

27

where it is possible to pass a tangent line from the effectiveness or probability curve through the origin, a maximum expected value exists at the point of tangency (sometimes referred to as the maximum marginal utility concept). It is obvious that this latter technique results in solutions (n_1) which are a good deal more sensitive to uncertainties in the resource than the technique under discussion. It is the author's feeling that the marginal utility criterion is only valid when there is no concern with uncertainty or when cost is of overriding importance. Our proposed technique always provides a higher value of n and effectiveness than the maximum expected value point. For the type of distribution illustrated in Fig. 3 it is probably a corrected observation that people tend to, or would like to, operate a system at a higher n than the maximum efficiency point (n_1).

As this type of distribution smooths out and goes over into the one originally illustrated in Fig. 1, the marginal utility concept falls apart and provides n = 0 as the best operating point, whereas the proposed technique still provides reasonable solutions. There are many real world situations where the degree of pessimism or optimism used in choosing values for the system parameters results in curves exhibiting this contrast in shape. In other words, if one is sufficiently pessimistic, a curve which produces a maximum efficiency or expected value point for an n or c greater than zero can be produced. This is to say that the maximum marginal utility concept is not one that is generally applicable to systems of this type. It is only applicable if sufficient pessimism is used in treating the particular physical system. One must be at least philosophically concerned about applying a criterion which breaks down over the range of uncertainties governing the input parameters to the system. In contrast, the technique we have proposed does not appear to suffer from this problem by providing reasonable answers regardless of how much optimism or pessimism is used in choosing the input parameters. Philosophically, therefore, it would appear to satisfy the meaning of a general criterion or technique to a better degree than the maximum marginal utility concept.

In Fig. 4, we plot a demand curve version of this type of distribution using a Fermi-Dirac distribution as the example. The rather obvious sensitivity of the maximum expectation point in comparison to the tradeoff point is of interest.

The following table is a summary of the various arguments that have been presented.

Our approach to modeling intuition is perhaps a simple one from a mathematical point of view; however, it is doubtful that any reasonable model of intuition or judgment involves sophisticated mathematics. In this case the model lends itself to such a rather diverse set of interpretations that one is led to an intuitive suspicion that it is in fact a valid model for the situation described.

Subject	Functions		Criterion	Value Function
Probability (Macro)	probability of success $P(n)$	expected value $\frac{NP(n)}{n}$	equal marginal return $\frac{dE}{dn} = \frac{dV}{dn}$	maximize $E-V$
Probability (Micro)	number of trials for a success $s = \frac{1}{P(n)}$	cost of a success $C = ns$	$\frac{dC}{dn} = -\frac{dV_c}{dn}$	minimize $C+V_c$
Uncertainty	maintain fixed n $n = \frac{N}{T}$	worst uncertainty $\frac{\delta N}{N} = -\frac{\delta T}{T}$	retain constant return $\delta E = 0$	$E = TP\left(\frac{N}{T}\right)$
Psychology	expectancy $P(n)$	incentive $\frac{N}{n} P(n)$	maximize motivation $\frac{dP}{dn} = \frac{P}{2n}$	maximize $\frac{N}{n} P(n)^2$
Economics	price $p(x)$	quantity $q(x)$	elasticity $= 1$ $-\frac{q}{p}\frac{dp}{dq} = 1$	maximize pq
General	effectiveness of the system $\mathcal{E}(c)$	efficiency $\frac{\mathcal{E}(c)}{c}$	average of marginal & marginal of average $\frac{1}{c}\frac{d\mathcal{E}}{dc} + \frac{d}{dc}\left(\frac{\mathcal{E}}{c}\right) = 0$	maximize $\frac{\mathcal{E}^2}{c}$

The curve is labeled with the following annotations:

$n = \infty$

Compromise point $n_o = 6.5$

Maximum expected value $n_o = 5.5$

$$P(n) = 1 - \frac{1}{e^{n - n_1} + 1}$$

$$n_1 = 4$$

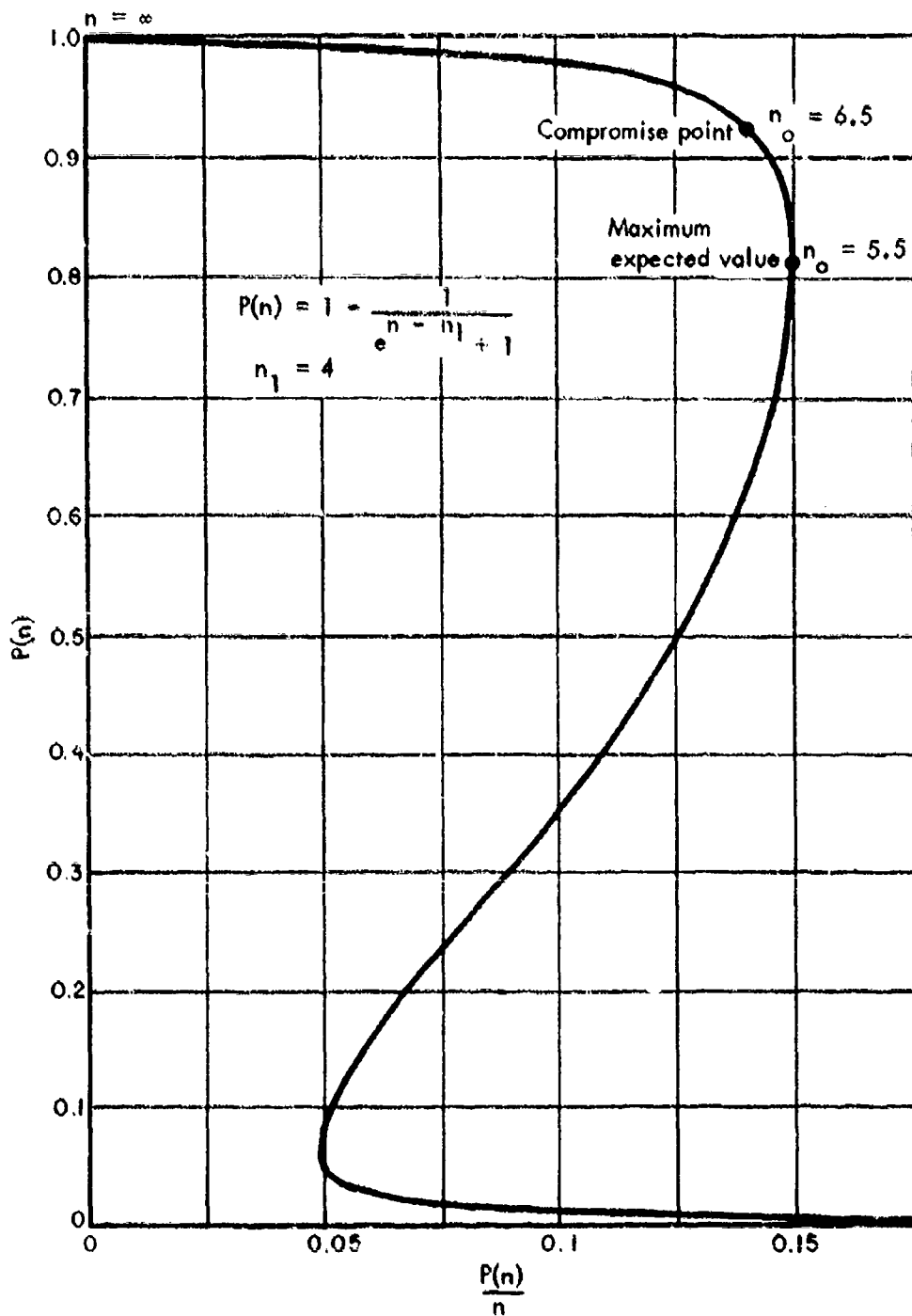Vertical axis: $P(n)$

Horizontal axis: $\dfrac{P(n)}{n}$

FIGURE 4 Demand Curve Comparison for Maximum Expected Value

VI. EXAMPLES

The examples taken are fairly simple in nature but provide a reasonable idea of the behavior of this technique as applied to different situations.

The first example is

$$P(n) = 1 - P_c^n \tag{60}$$

which is, for example, the kill probability for n independent shots at a target or the probability of a success in n trials when P_0 is the probability of no success in one trial.

Applying the criterion of maximizing $\dfrac{P^2(n)}{n}$, we obtain

$$x \left(1 + 2\ln \frac{1}{x}\right) = 1 \tag{61}$$

where

$$x = P_0^n \tag{62}$$

This results in

$$P(n_0) = 0.71533 \tag{63}$$

$$P_0^{n_0} = 0.28467 \tag{64}$$

Therefore, for this type of probability function our tradeoff point is always a probability of 0.715 and this provides us with a unique relation between P_0 and n_0 (tradeoff point). Usually P_0 can be a function of a number of system variables and an examination can now be made of these variables versus n_0. In Fig. 5 we plot the interesting functions for a particular P_0 and indicate the tradeoff point.

31

FIGURE 5 Example i

32
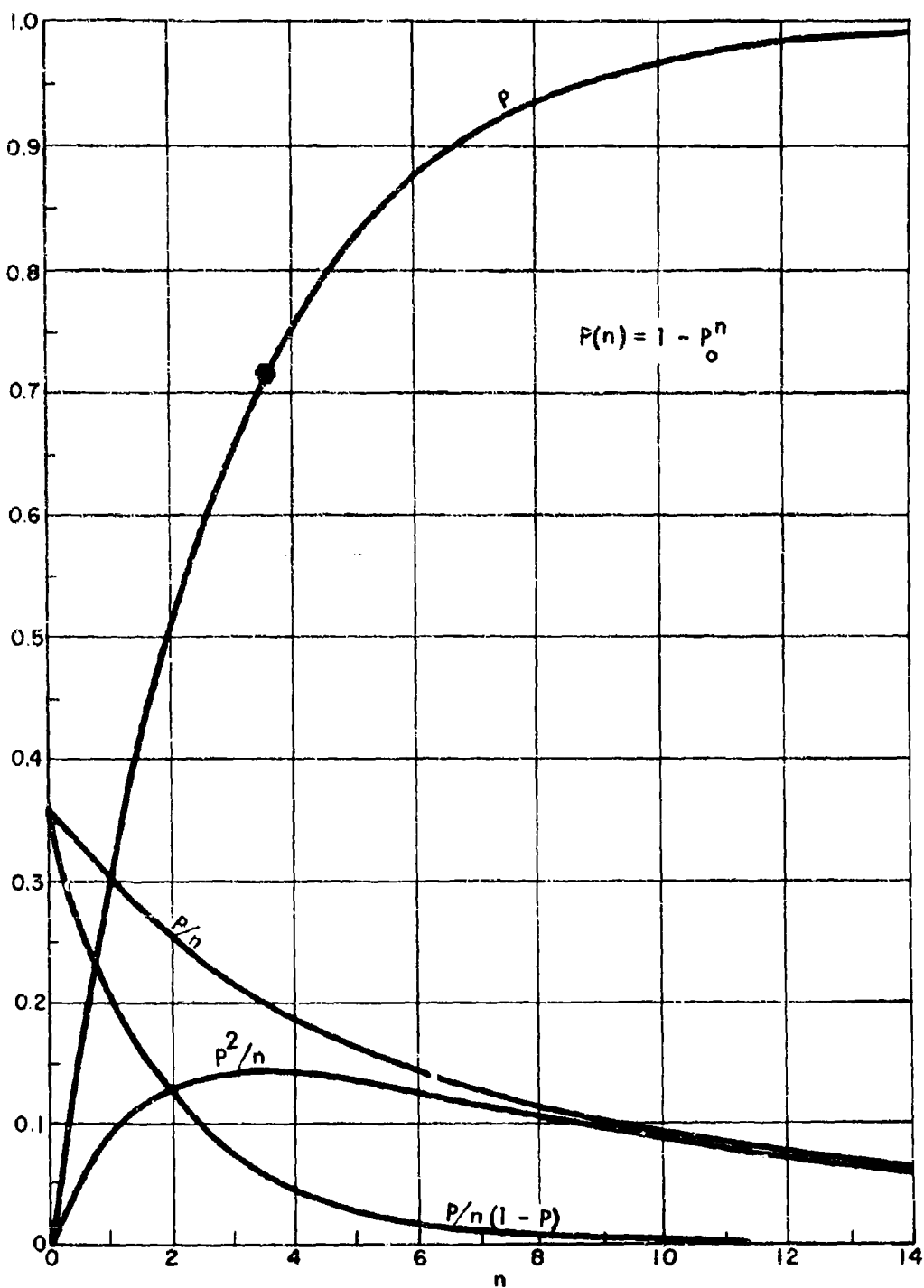
Let us suppose an experimenter is conducting a set of experiments under the following conditions.

(1) His object in each experiment is to observe and make measurements on a single event.

(2) The event will occur only once per experiment; however, the experiment must be preset to run a fixed time, t.

(3) There is a known lifetime in occurrence of the event, τ, such that the probability of occurrence is

$$P(t) = 1 - e^{-\frac{t}{\tau}} \qquad (65)$$

as a function of elapsed time, t.

(4) To get good statistics on his measurements he wants to make as many of these experiments as possible. However, he feels the total time he'll be able to operate the experimental setup is limited; he does not know the exact limit.

Equation 65 is of the same functional form as 60 and, therefore, applying the criteria, we find the experimenter should aim for a 0.715 probability of success. From Eq. 64 we may further deduce that the time he should spend on one experimental setup is

$$t_o = 1.26\tau \qquad (66)$$

There are also analogies one can draw in the quality control or inspection process. Another example might be that if τ were the average time it took a salesman to make a sale, we would expect t_o to be the amount of time he should reserve to devote to any one customer.

Another example is a probability function of the type

$$P(n) = 1 - e^{-k\sqrt{n}} \left(1 + k\sqrt{n}\right) \qquad (67)$$

This is an approximate solution for the fraction of damage against an area target with a Gaussian distributed value function and where

33

weapons of small lethal area are delivered randomly.* Application of our criteria results in

$$e^{-x}(1 + x + x^2) = 1 \tag{68}$$

where

$$x = k\sqrt{n_0} \tag{69}$$

Therefore

$$k\sqrt{n_0} = 1.7933 \tag{70}$$

and

$$P(n_0) = 0.535 \tag{71}$$

As one would suspect, since Eq. 67 increases slower than 60 as a function of n, the tradeoff point produces a lower probability of success. This example is illustrated in Fig. 6.

An example that particularly illustrates the behavior of this technique is a probability of success that is governed by a Gaussian density function of mean, m, and standard deviation, σ

$$P(n) = \int_{-\infty}^{n} \frac{e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}\,dx \tag{72}$$

Applying the criteria we find

$$2\left(\frac{n}{\sigma}\right)\frac{e^{-\frac{1}{2}\left(\frac{n-m}{\sigma}\right)^2}}{\sqrt{2\pi}} = \int_{-\infty}^{\frac{n-m}{\sigma}} \frac{e^{-\frac{1}{2}x^2}dx}{\sqrt{2\pi}} \tag{73}$$
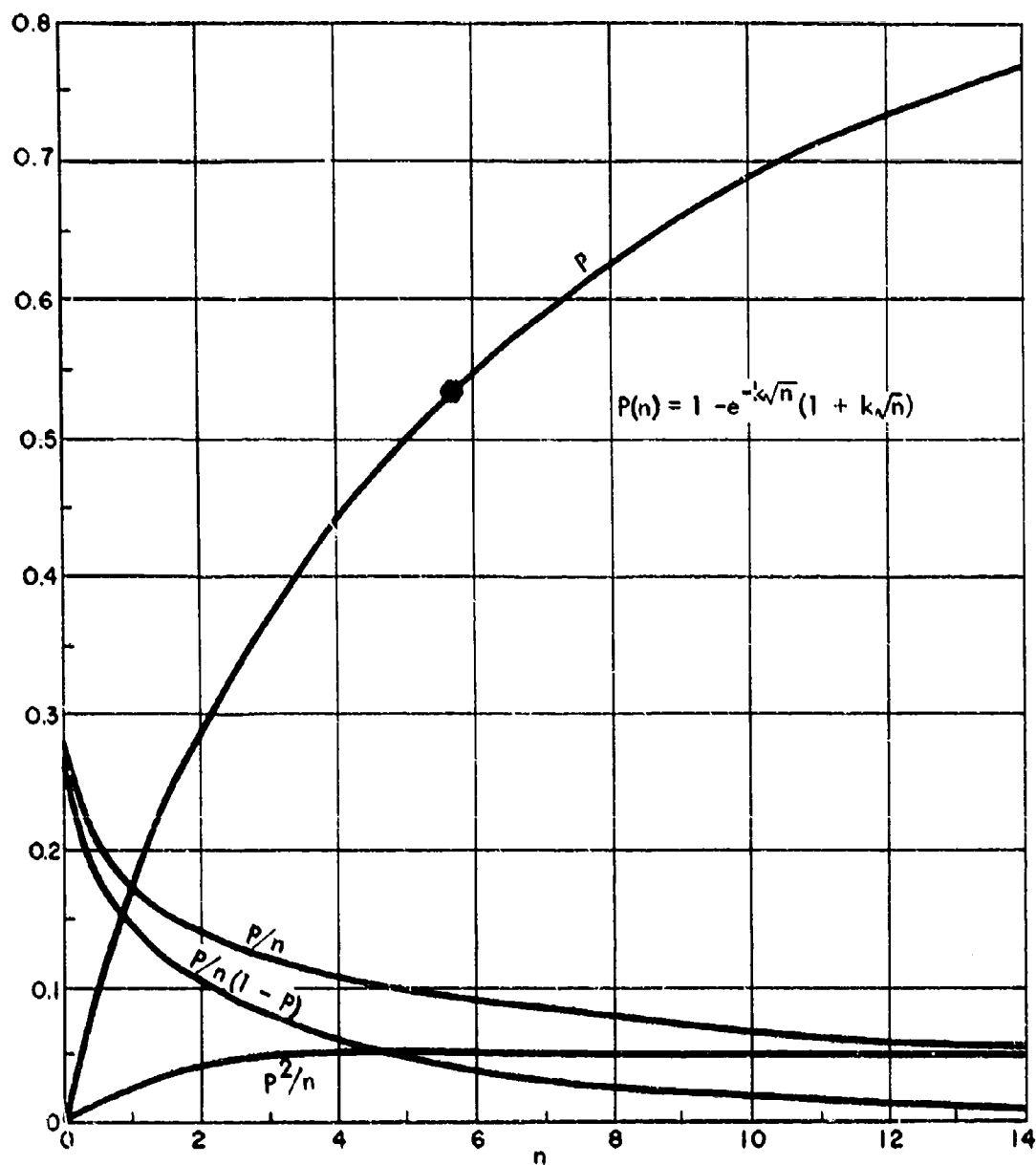
_____

*Results due to Hugh Everett of Lambda Corporation.

34

$$P(n) = 1 - e^{-k\sqrt{n}}(1 + k\sqrt{n})$$

FIGURE 6 Example 2

35

In Fig. 7 we plot the solution curve for n/σ versus m/σ and the resulting probability P(n/σ) as a function of m/σ. Since the original probability function has an inflection point, it is not surprising that there are two solutions for n/σ as a function of m/σ. What is of particular interest is that if m/σ < 0.5, there appears to be no equilibrium point between the two functions. The exact value of the cutoff point on the curve is

$$\frac{m}{\sigma} = 0.5, \quad \frac{n}{\sigma} = 1, \quad P\left(\frac{n}{\sigma}\right) = 0.6915 \tag{74}$$
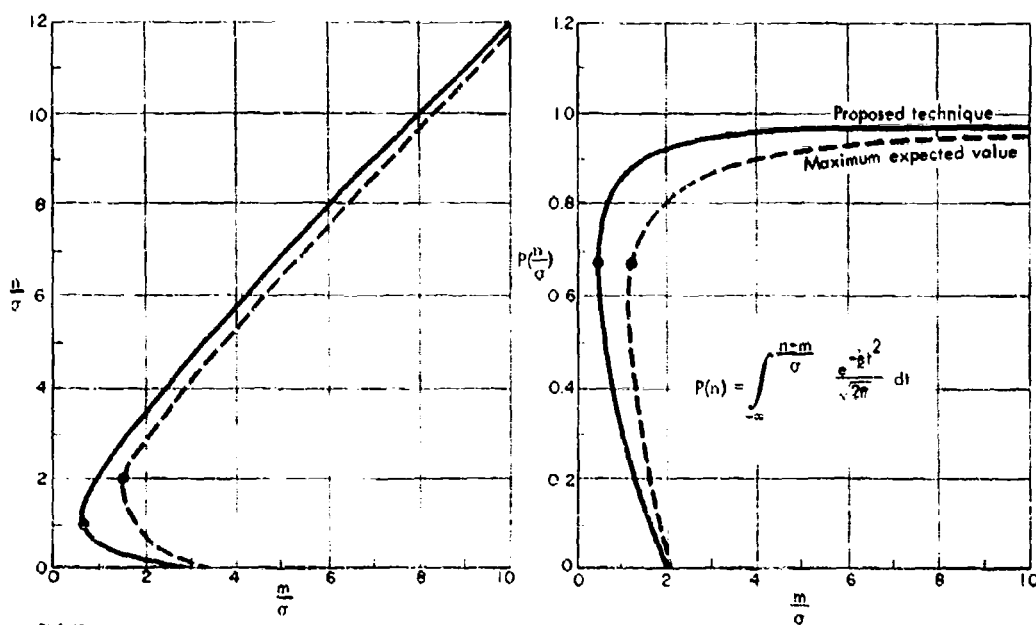

FIGURE 7 Example 3

Also plotted for contrast is the solution for maximum expected value since this is the type of distribution which exhibits this property. The cutoff point in this case is

$$\frac{m}{\sigma} = 1.5, \quad \frac{n}{\sigma} = 2, \quad P\left(\frac{n}{\sigma}\right) = 0.6915 \tag{75}$$

The solutions below these cutoff points are spurious in the sense that they represent a relative minimum with respect to a boundary

36

maximum existing at $\frac{n}{\sigma} = 0$. Therefore, below the cutoff point the solution is always the smallest allowable n. We see, for example, that at $\frac{m}{\sigma} = 2$ the difference in effectiveness for the two curves is 0.13 ($P(n) = 0.93$ as compared to $P(n) = 0.8$). The resulting difference in expected return is less than 10 percent.

In general, the numerical results of these examples do not appear to contradict our initial premises. On an intuitive basis, at least to the author, the resulting numerical values would seem to provide a point of equality or indifference in relation to overall expected value and individual probability of success.

VII. CONCLUDING REMARKS

It is certainly true that in the real world we are unlikely to encounter situations where an equality of importance exists between two opposing goals. Usually, in any given situation a strong bias exists for one goal over the other. In the case of a very strong bias, I don't believe there is any disagreement that our technique provides a boundary point on the region to be considered. The decision maker in his role has the option to draw a narrower region of consideration if it is his intuitive opinion that his bias is strong enough to warrant this. However, the analyst is often forced to set up his own boundaries in at least subsections of the general problem under consideration. Right now this is often an intuitive or arbitrary process on the part of the analyst. In some sense this seems to be a contradiction between the role of the decision maker and the analyst. Ideally at least, it does not seem philosophically correct that the analyst should be called upon to make arbitrary decisions—decisions not reproducible between two separate analysts acting independently. The point, of course, is that in the case of strong bias an analytical tool exists for setting a boundary; therefore, it should be employed by the analyst in such cases.

The point of question or disagreement arises when the bias is weak. As the bias weakens, the region of interest becomes very hazy as to any sort of fixed limits. If it is granted that in a question of the sort considered here strong bias one way drives one to a completely opposite conclusion from strong bias the other way, then it follows that in the case of no bias there should exists a point of compromise between the two goals. Furthermore, the fundamental concepts in the fields of psychology and statistics would suggest that

38

if a great many humans are presented with exactly the same compromise situation, then the number of humans choosing a particular point of compromise versus the possible choices will take on a Gaussian shape about some mean value. This would then imply that there exists a mean that is unique to any particular compromise situation. It would then seem self-evident that this point is the natural reference about which to examine the consequences of various alternatives when only a weak bias exists for one goal over the other. Although he would like to, the author cannot infer that such a mean found by experimentation is indeed the analytically determined point derived in this paper; but he would suggest that this is indeed possible. At least, there would seem to be enough pleasant analytical properties of this point to warrant its consideration. It is true, however, that the mean and its variance derived in this experimental manner would, in all likelihood, be a strong function of the characteristics of the human subjects involved. The decision point derived would, hopefully, be somewhat different between the use of laymen or the use of decision makers as a subject class.

In addition, an alternative limiting argument which should be considered is the proposition that if, indeed, in the case of strong bias for one of the two goals, the boundary or limit point derived here is universally "acceptable" to all concerned, then the inference can be made that a boundary point that is unique (one point), regardless of for which goal the strong bias exists, must be the mean compromise point between the two goals.

Perhaps, in conclusion, the author should explain his interpretation of the quote by Gibran that begins this report. The problem the author has treated is open to arbitrary solution. One may choose an arbitrary operating point based upon intuition, or one may arbitrarily choose a criterion, such as a confidence limit or maximum marginal utility, which produces a unique point but in no less an arbitrary manner. It is the author's problem that he does not see or "understand" any inherent "truth" in any technique or choice applied

39

to this type of problem other than the ability of rational men to make a reasonable judgment. The author's attempt has been to provide another tool as a useful guide to this intuitive process and to offer explanation as to the nature and properties of this tool.

REFERENCES

1. G. M. Becker and C. G. McClintock, "Value: Behavioral Decision Theory," *Annual Review of Psychology*, in press (1967).

2. W. Edwards, "Behavioral Decision Theory," *Annual Review of Psychology*, Vol. 12, p. 473 (1961).

3. W. J. Baumel, *Economic Theory and Operations Analysis*, Prentice-Hall (1961).

4. J. W. Atkinson, "Motivational Determinants of Risk Taking Behavior," *Psychological Review*, Vol. 64, No. 6 (1957).

5. J. Cohen, "Subjective Probability," *Scientific American*, Vol. 197, p. 128 (November 1957).

41

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Institute for Defense Analyses	Unclassified
	2b. GROUP None

3. REPORT TITLE

A Heuristic Compromise Between Probability of Success and Limited Resources

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research Paper P-338   August 1967

5. AUTHOR(S) (First name, middle initial, last name)

Murray Turoff

6. REPORT DATE	7a. TOTAL NO OF PAGES	7b. NO OF REFS
August 1967	41	5

8a. CONTRACT OR GRANT NO	9a. ORIGINATOR'S REPORT NUMBER(S)
DAHC15 67 C 0011	P-338
b. PROJECT NO T-28	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	NA

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	13. SPONSORING MILITARY ACTIVITY
NA	NA

13. ABSTRACT

This paper proposes an analytical technique for modeling by the analyst of a particular decision situation. Evidence supporting the basic proposal, properties of the technique, discussions of its possible applications, and examples are also presented.

The basic problem is a compromise between the probability of success in a single trial versus the total expected number of successes over many trials under the constraint of a limited and uncertain resource. The analogous situation of choosing an operating point for a system which is a compromise between the effectiveness of the system and its efficiency (effectiveness divided by cost) is also discussed.

DD FORM 1473 NOV 65

K.Y WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT